Random walk with heavy tail and negative drift conditioned by its minimum and final values

Vincent Bansaye, Vladimir Vatutin

Random walk with heavy tail and negative drift conditioned by its minimum and final values

Vincent Bansaye* and Vladimir Vatutin†

December 11, 2013

Abstract

We consider random walks with finite second moment which drifts to $-\infty$ and have heavy tail. We focus on the events when the minimum and the final value of this walk belong to some compact set. We first specify the associated probability. Then, conditionally on such an event, we finely describe the trajectory of the random walk. It yields a decomposition theorem with respect to a random time giving a big jump whose distribution can be described explicitly.

1 Introduction and main results

We consider a random walk $\mathbf{S} = (S_n : n \geq 0)$ generated by a sequence $(X_n : n \geq 1)$ of i.i.d. random variables distributed as a random variable X. Thus,

$$S_n = \sum_{i=1}^{n} X_i, \qquad (S_0 = 0).$$

We assume that the random walk has a negative drift

$$\mathbf{E}[X] = -a < 0. \tag{1}$$

and a heavy tail

$$A(x) = \mathbf{P}(X > x) = \frac{l(x)}{x^\beta}, \tag{2}$$

where $\beta > 2$ and $l(x)$ is a function slowly varying at infinity. Thus, the random variable X under the measure $\mathbf{P}$ does not satisfy the Cramer condition and has finite variance. We further suppose that, for any fixed $\Delta > 0$,

$$x\left[\frac{l(x+\Delta)}{l(x)} - 1\right] \overset{x\to\infty}{\longrightarrow} 0$$

*CMAP, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France; e-mail: bansaye@polytechnique.edu

†Department of Discrete Mathematics, Steklov Mathematical Institute, 8, Gubkin str., 119991, Moscow, Russia; e-mail: vatutin@mi.ras.ru

which is equivalent to

$$\mathbf{P}(X \in (x, x+\Delta]) = \frac{\Delta\beta\mathbf{P}(X > x)}{x}(1+o(1)) = \frac{\Delta\beta A(x)}{x}(1+o(1)) \qquad (3)$$

as $x \to \infty$.

To formulate the results of the present paper we introduce two important random variables

$$M_n = \max(S_1, \dots, S_n), \quad L_n = \min(S_1, \dots, S_n)$$

and two right-continuous functions $U : \mathbb{R} \to \mathbb{R}_0 = \{x \geq 0\}$ and $V : \mathbb{R} \to \mathbb{R}_0$ given by

$$U(x) = 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \leq x, M_k < 0), \quad x \geq 0,$$

$$V(x) = 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k > x, L_k \geq 0), \quad x \leq 0,$$

and 0 elsewhere. In particular $U(0) = V(0) = 1$. It is well-known that $U(x) = O(x)$ for $x \to \infty$. Moreover, $V(-x)$ is uniformly bounded in x in view of $\mathbf{E}X < 0$.

It will be convenient to write $\mathbf{1}_n$ for the $n-$dimensional vector whose coordinates are all equal to 1 and set $\mathbf{S}_{j,n} = (S_j, S_{j+1}, \cdots, S_n)$ if $j \leq n$ with $\mathbf{S}_n = \mathbf{S}_{0,n}$ and $\mathbf{S}_{n,0} = (S_n, S_{n-1}, \cdots, S_0)$. Similar notation will be used for nonrandom vectors. Say, $\mathbf{s}_{n,0} = (s_n, s_{n-1}, \cdots, s_0)$. Let

$$b_n = \beta \frac{\mathbf{P}(X > an)}{an}.$$

With this notation in hands, we first describe the asymptotic behavior of the probability of the event that the random walk remains within the time interval $[0, n]$ above some level $-x$ and ends up at time n below the level T.

Theorem 1 *For any $x \geq 0$ and $T > -x$, as $n \to \infty$,*

$$\mathbf{P}(S_n < T, L_n \geq -x) \sim b_n U(x) \int_0^{x+T} V(-z)\, dz \qquad (4)$$

and for any $x \geq 0$ and $T < x$, as $n \to \infty$,

$$\mathbf{P}(S_n > T, M_n < x) \sim b_n V(-x) \int_0^{x-T} U(z)\, dz\,. \qquad (5)$$

Our second goal is to demonstrate that if the event $\{S_n < T, L_n \geq -x\}$ occurs then the trajectory of the random walk on $[0, n]$ has a big jump of the order $an + O(\sqrt{n})$, such a jump is unique and happens at the beginning of the trajectory. Using this fact we also describe the full trajectory of the random walk.

Theorem 2 *For all $x > 0$ and $T \in \mathbb{R}$ there exists a sequence of numbers $\pi_j = \pi_j(x) > 0$, $\sum_{j\geq 0} \pi_j = 1$, such that for each j the following properties hold:*

(i) $\lim_{n\to\infty} \mathbf{P}\left(X_j \geq an/2 | L_n \geq -x,\ S_n \leq T\right) = \pi_j$;

(ii) For each measurable and bounded function $F : \mathbb{R}^j \to \mathbb{R}$ and each family of measurable uniformly bounded functions $F_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$\lim_{\varepsilon\to 0} \sup_{n\in\mathbb{N}, \mathbf{s}_n\in\mathbb{R}^{n+1}} |F_n(\mathbf{s}_n + \epsilon \mathbf{1}_{n+1}) - F_n(\mathbf{s}_n)| = 0, \tag{6}$$

we have as $n \to \infty$

$$\mathbf{E}\left[F(\mathbf{S}_{j-1}) F_{n-j}(\mathbf{S}_{j,n}) | L_n \geq -x,\ S_n \leq T,\ X_j \geq an/2\right]$$
$$-\mathbf{E}\left[F(\mathbf{S}_{j-1}) | L_{j-1} \geq -x\right] \mathbf{E}_\mu\left[F_{n-j}(\mathbf{S}'_{n-j,0}) | L'_\infty \geq -x\right] \to 0,$$

where $\mathbf{S}'$ is a random walk with step $-X$ and positive drift, L'_∞ is its global minimum and μ is a probability measure given by :

$$\mu(dy) = 1_{y\in[-x,T]} \theta^{-1} \mathbf{P}_y(L'_\infty \geq -x) dy, \quad \theta = \int_{-x}^{T} dy \mathbf{P}_y(L'_\infty \geq -x). \tag{7}$$

In words, this theorem yields the decomposition of the trajectory of $(S_i : i \leq n)$ conditioned by its minimum L_n and final value S_n. It says that conditionally on $L_n \geq -x$ and $S_n = s$, $\mathbf{S}$ jumps with probability π_j at some (finite) time j. Before this time, $\mathbf{S}$ is simply conditioned to be larger than $-x$. After this time, reversing the trajectory yields a random walk $\mathbf{S}'$ (with positive drift) conditioned to be larger than $-x$. The size of the jump at time j links the value S_{j-1} to $S'_{n-j-1} = s + a(n-j-1) + \sqrt{n} W_n$, where, thanks to the central limit theorem, W_n converges in distribution, as $n \to \infty$ to a Gaussian random variable. Thus this big jumps is of order $an + W_n\sqrt{n}$, as stated below. The proof is differed to Section 5.

Corollary 3 *Let $\varkappa = \inf\{j \geq 1 : X_j \geq an/2\}$. Under $\mathbf{P}$, conditionally on $L_n \geq -x$ and $S_n \leq T$, $\varkappa$ converges in distribution to a proper random variable whose distribution $(\pi_j : j \geq 1)$ is specified by*

$$\pi_j = \pi_j(x) = \frac{\mathbf{P}(L_j \geq x)}{\sum_{k\geq 0} \mathbf{P}(L_k \geq x)}$$

and

$$\frac{X_\varkappa - an}{\sqrt{n}}$$

converges in distribution to a centered Gaussian law with variance $\sigma^2 = Var(X)$.

Proof. The expression of π_j can be found in STEP 4 of the proof of Theorem 2, see (23). The second part of the corollary is an application of the second part of the mentioned theorem with

$$F(s_0, \cdots, s_j) = 1, \qquad F_n(s_1, ..., s_{n+1}) = g((s_1 - an)/\sqrt{n})$$

for g uniformly continuous and bounded if one takes into account the positivity of the drift of $\mathbf{S}'$ allowing to neglect the condition $L'_\infty \geq -x$ and to use the central limit theorem. ■

We note that random walks with negative (or positive) drift satisfying conditions (1) and (2) (and even weaker assumptions) have been investigated by many authors (see, for instance, [6, 11, 14, 15] and monograph [9] with references therein. Article [11] is the most close to the subject of the present paper. Durrett has obtained there scaled limit results for the random walk meeting conditions (1) and (2) but conditionally on the minimum value only. He has shown that, for each $M \geq a$ the size of the big jump may exceed the value Mn with a positive probability. The additional condition on the final value we impose in Theorem 1 modifies the size of the big jump by forcing it to be concentrated in a vicinity of point an with deviation of order $\sqrt{n}$ and allows us to provide in Theorem 2 a non scaled decomposition of the asymptotic conditional path. We stress that when the increments of the random walk are in the domain of attraction of a Gaussian law with zero mean, such problems have been investigated. See, in particular, [16] for the convergence of the scaled random walk to the Brownian excursion.

The initial motivations to get the results of the present paper come from branching processes in random environment (BPRE). The survival probability of such processes are deeply linked to the behavior of the random walk associated with the successive log mean offspring [13, 3, 4, 1, 2, 8, 17], namely $\log m$. The fine results given here are required to get the asymptotic survival probability of the subcritical class of BPRE such that this $\log m$ has density

$$p\left(x\right) = \frac{l_0(x)}{x^{\beta+1}} e^{-\rho x}, \tag{8}$$

where $l_0(x)$ is a function slowly varying at infinity, $\beta > 2$, $\rho \in (0,1)$. We refer to [5] for precise statements and proofs.

2 Preliminaries : Some classical results on random walks

Our arguments essentially use a number of statements from the theory of random walks, that are included into this section.
In the sequel we shall meet the situations in which the random walk starts from any point $x \in \mathbb{R}$. In such cases we write for probabilities as usual $\mathbf{P}_x\left(\cdot\right)$. We use for brevity $\mathbf{P}$ instead of $\mathbf{P}_0$.

We define

$$\tau_n = \min\left\{0 \leq k \leq n : S_k = \min(0, L_n)\right\}, \ \tau = \min\left\{k > 0 : \ S_k < 0\right\}$$

and let

$$D=\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{P}\left(S_k\geq 0\right).$$

Now we list some known statements for the convenience of references. The first lemma is directly taken from [9], Theorems 8.2.4, page 376 and 8.2.18, page 389.

Lemma 4 *Under conditions (1) and (2), as* $n\to\infty$

$$\mathbf{P}\left(L_n\geq 0\right)=\mathbf{P}\left(\tau>n\right)\sim e^{D}\mathbf{P}\left(X>an\right) \tag{9}$$

and for any fixed $x>0$

$$\lim_{n\to\infty}\frac{\mathbf{P}\left(L_n\geq -x\right)}{\mathbf{P}\left(\tau>n\right)}=U(x). \tag{10}$$

The next statement is an easy corollary of Theorem 4.7.1, page 218 of monograph [9].

Lemma 5 *Let X be a non-lattice random variable with $\mathbf{E}\left[X\right]=-a<0$ whose distribution satisfies condition* (3)*. If $\tilde{S}_n=X_1+\cdots+X_n+an$, then for any $\Delta>0$ uniformly in $x\geq n^{2/3}$,*

$$\mathbf{P}\left(\tilde{S}_n\in[x,x+\Delta)\right)=\frac{\Delta\beta nA(x)}{x}(1+o(1)).$$

In the sequel we use several times the following lemma, in which i) does not require a proof and ii) is a special case of Theorem 1 in [10].

Lemma 6 *Let (r_n) be a regularly varying sequence with $\sum_{k=0}^{\infty}r_k<\infty$.*
i) If $\delta_n\sim dr_n$, $\eta_n\sim er_n$, then $\sum_{i=0}^{n}\delta_i\eta_{n-i}\sim cr_n$ with $c=d\sum_{k=0}^{\infty}\eta_k+e\sum_{k=0}^{\infty}\delta_k$ as $n\to\infty$.
ii) If $\sum_{k=0}^{\infty}\alpha_k t^k=\exp\left(\sum_{k=0}^{\infty}r_k t^k\right)$ for $|t|<1$, then $\alpha_n\sim cr_n$ with $c=\sum_{k=0}^{\infty}\alpha_k$ as $n\to\infty$.

We introduce two functions

$$\begin{aligned}K_1\left(\lambda\right)&=\frac{1}{\lambda}\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{E}\left[e^{\lambda S_n};S_n<0\right]\right\}\\&=\frac{1}{\lambda}\left(1+\sum_{n=1}^{\infty}\mathbf{E}\left[e^{\lambda S_n};M_n<0\right]\right)=\int_0^{\infty}e^{-\lambda x}U(x)dx,\end{aligned} \tag{11}$$

$$\begin{aligned}K_2\left(\lambda\right)&=\frac{1}{\lambda}\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{E}\left[e^{-\lambda S_n};S_n\geq 0\right]\right\}\\&=\frac{1}{\lambda}\left(1+\sum_{n=1}^{\infty}\mathbf{E}\left[e^{-\lambda S_n};L_n\geq 0\right]\right)=\int_0^{\infty}e^{-\lambda z}V(-z)dz,\end{aligned} \tag{12}$$

being well defined for $\lambda > 0$.

Note that the intermediate equalities in (11) and (12) are simply versions of the Baxter identities (see, for instance, Chapter XVIII.3 in [12] or Chapter 8.9 in [7]).

3 Asymptotic behavior of the distribution of (S_n, L_n)

Basing on the three previous lemmas, we prove the following statement.

Lemma 7 *Assume that* $\mathbf{E}[X] < 0$ *and that* $A(x)$ *meets condition* (3). *Then, for any* $\lambda > 0$ *as* $n \to \infty$

$$\mathbf{E}\left[e^{\lambda S_n}; \tau_n = n\right] = \mathbf{E}\left[e^{\lambda S_n}; M_n < 0\right] \sim K_1(\lambda)\, b_n \tag{13}$$

and

$$\mathbf{E}\left[e^{-\lambda S_n}; \tau > n\right] = \mathbf{E}\left[e^{-\lambda S_n}; L_n \geq 0\right] \sim K_2(\lambda)\, b_n. \tag{14}$$

Proof. We prove (14) only. Statement (13) (proved in [17] under a bit stronger conditions) may be checked in a similar way. First we evaluate the quantity

$$\mathbf{E}\left[e^{-\lambda S_n}; S_n \geq 0\right] = \mathbf{E}\left[e^{-\lambda S_n}; 0 \leq S_n < \lambda^{-1}(\beta+2)\log n\right] + O\left(n^{-\beta-2}\right). \tag{15}$$

Clearly, for any $\Delta > 0$

$$\begin{aligned}
&\sum_{0 \leq k \leq (\beta+2)\lambda^{-1}\Delta^{-1}\log n} e^{-\lambda(k+1)\Delta}\mathbf{P}\left(k\Delta + an \leq \tilde{S}_n \leq (k+1)\Delta + an\right) \\
&\quad \leq \mathbf{E}\left[e^{-\lambda S_n}; 0 \leq S_n < \lambda^{-1}(\beta+2)\log n\right] \\
&\quad \leq \sum_{0 \leq k \leq (\beta+2)\lambda^{-1}\Delta^{-1}\log n} e^{-\lambda k\Delta}\mathbf{P}\left(k\Delta + an \leq \tilde{S}_n \leq (k+1)\Delta + an\right).
\end{aligned}$$

Recall that by Lemma 5 in the range of k under consideration

$$\begin{aligned}
\mathbf{P}\left(k\Delta + an \leq \tilde{S}_n \leq (k+1)\Delta + an\right) &= \frac{\Delta\beta n}{(k\Delta + an)} A(k\Delta + an)(1 + o(1)) \\
&= \frac{\Delta\beta}{a} A(an)(1 + o(1)),
\end{aligned}$$

where $o(1)$ is uniform in $0 \leq k \leq (\beta+2)\lambda^{-1}\Delta^{-1}\log n$. Now passing to the limit as $n \to \infty$ we get

$$\begin{aligned}
\Delta\sum_{k=0}^{\infty} e^{-\lambda(k+1)\Delta} &\leq \liminf_{n\to\infty} \frac{a\mathbf{E}\left[e^{-\lambda S_n}; 0 \leq S_n < \lambda^{-1}(\beta+2)\log n\right]}{\beta A(an)} \\
&\leq \limsup_{n\to\infty} \frac{a\mathbf{E}\left[e^{-\lambda S_n}; 0 \leq S_n < \lambda^{-1}(\beta+2)\log n\right]}{\beta A(an)} \\
&\leq \Delta\sum_{k=0}^{\infty} e^{-\lambda k\Delta}.
\end{aligned}$$

Letting $\Delta \to 0+$, we see that

$$\lim_{n\to\infty} \frac{a\mathbf{E}\left[e^{-\lambda S_n}; 0 \le S_n < \lambda^{-1}(\beta+2)\log n\right]}{\beta A(an)} = \lambda^{-1}.$$

Combining this with (15) we conclude that, as $n \to \infty$

$$\mathbf{E}\left[e^{-\lambda S_n}; S_n \ge 0\right] \sim \frac{\beta}{a\lambda} A(an)(1+o(1)) \sim \frac{\beta}{a\lambda}\mathbf{P}(X > an). \tag{16}$$

We know by the Baxter identity (see, for instance, Chapter 8.9 in [7]) that for $\lambda > 0$ and $t \in [0,1]$

$$1 + \sum_{n=1}^{\infty} t^n \mathbf{E}\left[e^{-\lambda S_n}; L_n \ge 0\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{t^n}{n}\mathbf{E}\left[e^{-\lambda S_n}; S_n \ge 0\right]\right\}.$$

From (16) and point i) of Lemma 6 with $r_n = b_n$ we get for $n \to \infty$,

$$\mathbf{E}\left[e^{-\lambda S_n}; L_n \ge 0\right] \sim K_2(\lambda) \frac{\beta \mathbf{P}(X > an)}{an},$$

where $K_2(\lambda)$ is specified by (12). This gives statement (14) of the lemma. ∎

Lemma 8 *For $x \ge 0, \lambda > 0$ we have as $n \to \infty$:*

$$\mathbf{E}_{-x}[e^{\lambda S_n}\,;\, M_n < 0] \;\sim\; b_n V(-x)\int_0^\infty e^{-\lambda z}U(z)\,dz, \tag{17}$$

$$\mathbf{E}_x[e^{-\lambda S_n}\,;\, L_n \ge 0] \;\sim\; b_n U(x)\int_0^\infty e^{-\lambda z}V(-z)\,dz. \tag{18}$$

Proof. This proof follows the line for proving Proposition 2.1 in [1]. By the continuity theorem for Laplace transforms Lemmas 6 and 7 give for any $x \in [0,\infty)$ and $\lambda > 0$

$$b_n^{-1}\mathbf{E}[e^{\lambda S_n}; M_n < 0\,,\, S_n > -x] \;\to\; \int_0^x e^{-\lambda z}U(z)\,dz, \tag{19}$$

$$b_n^{-1}\mathbf{E}[e^{-\lambda S_n}; L_n \ge 0\,,\, S_n < x] \;\to\; \int_0^x e^{-\lambda z}V(-z)\,dz. \tag{20}$$

Further, using duality we have

$$\begin{aligned}
\mathbf{E}[e^{\lambda S_n}; M_n < x] \;&=\; \sum_{i=0}^{n-1}\mathbf{E}[e^{\lambda S_n}; S_0,\ldots,S_i \le S_i < x\,,\, S_i > S_{i+1},\ldots,S_n] \\
&\quad +\; \mathbf{E}[e^{\lambda S_n}; S_0,\ldots,S_n \le S_n < x] \\
&=\; \sum_{i=0}^{n-1}\mathbf{E}[e^{\lambda S_i}; L_i \ge 0, S_i < x]\cdot\mathbf{E}[e^{\lambda S_{n-i}}; M_{n-i} < 0] \\
&\quad + \mathbf{E}[e^{\lambda S_n}; L_n \ge 0, S_n < x].
\end{aligned}$$

This formula together with (13), (20), the left continuity of $V(-z)$ for $z > 0$ implying $V(0) = V(0-) = 1$, and the equations

$$1+\sum_{k=1}^{\infty}\mathbf{E}[e^{\lambda S_k}; L_k \geq 0, S_k < x]$$

$$= 1+\int_{(0,x)} e^{\lambda z}\, dV(-z) \;=\; e^{\lambda x}V(-x)-\lambda\int_0^x e^{\lambda z}V(-z)\, dz\ ,$$

$$1+\sum_{k=1}^{\infty}\mathbf{E}[e^{\lambda S_k}; M_k < 0] \;=\; \lambda\int_0^{\infty} e^{-\lambda z}U(z)\, dz$$

yield by Lemma 6 i) that for $\lambda > 0$ and $x > 0$

$$b_n^{-1}\mathbf{E}[e^{\lambda S_n}; M_n < x] \;\to\; V(-x)e^{\lambda x}\int_0^{\infty} e^{-\lambda z}U(z)\, dz,$$

which gives (17) by multiplying by $\exp(-\lambda x)$. Using similar arguments one can get (18). ■

The continuity theorem for Laplace transforms and (17) and (18) yield the asymptotic distribution of (S_n, L_n) on compacts sets.

Proof of Theorem 1. By (18) and the continuity theorem for Laplace transforms for any $x \geq 0$ and $y > x$ we have

$$\mathbf{E}_x[e^{-\lambda S_n}\ ;\ S_n < y, L_n \geq 0] \;\sim\; b_n U(x)\int_0^y e^{-\lambda z}V(-z)\, dz$$

giving

$$\mathbf{P}_x\left(S_n < y, L_n \geq 0\right) \sim\; b_n U(x)\int_0^y V(-z)\, dz$$

or

$$\mathbf{P}\left(S_n < y-x, L_n \geq -x\right) \sim\; b_n U(x)\int_0^y V(-z)\, dz$$

justifying (4).

The asymptotic representation (5) may be checked by the same arguments.

■

4 Conditional description of the random walk

In this and subsequent sections we agree to denote by $C, C_1, C_2, ...$ positive constants which may be different in different formulas or even within one and the same complicated expression.

Our first result shows that the random walk may stay over a fixed level for a long time only if it has at least one big jump. Let

$$\mathcal{B}_j\left(y\right) = \{X_j + a \leq y\}, \qquad \mathcal{B}^{(n)}\left(y\right) = \cap_{j=1}^{n}\mathcal{B}_j\left(y\right).$$

Lemma 9 *If $\mathbf{E}[X] = -a < 0$ and condition (3) is valid then there exists $\delta_0 \in (0, 1/4)$ such that for all $\delta \in (0, \delta_0)$, $k \in \mathbb{Z}$, and $an/2 - u \geq M$,*

$$\mathbf{P}_u(\max_{1\leq j\leq n} X_j \leq \delta an,\ S_n \geq k) \leq \varepsilon_M(k) n^{-\beta-1}, \quad \textit{where} \qquad \varepsilon_M(k) \downarrow_{M\to\infty} 0.$$

Proof. Set $Y_n = (S_n + an)/\sigma$ where $\sigma^2 = Var(X)$ and $S_0 = 0$. It follows from Theorem 4.1.2 and Corollary 4.1.3 (i) in [9] (see also estimate (4.7.7) in the mentioned book) that if $r > 2$ and $\delta > 0$ are fixed then for $x \geq n^{2/3}$ and all sufficiently large n

$$\mathbf{P}(\mathcal{B}^{(n)}(x\sigma r^{-1}), Y_n \geq x) \leq \left[n\mathbf{P}(X + a \geq \sigma x r^{-1})\right]^{r-\delta}.$$

Since $l(x)$ in (2) is slowly varying, $x^{-1/4} l(x) \to 0$ as $x \to \infty$. Hence we get for all sufficiently large n and $\beta > 2$

$$\left[n\mathbf{P}(X + a \geq \sigma x r^{-1})\right]^{r-\delta} \leq C\left(\frac{nl(x)}{x^\beta}\right)^{r-\delta} \leq C\left(\frac{n}{x^{\beta-1/4}}\right)^{r-\delta} \leq C\left(\frac{1}{n^{1/6}}\right)^{r-\delta}.$$

We fix now $r > 2, \delta_0 < 1/4$ with $r\delta_0 = 1/2$ so that $(r-\delta)/6 > \beta + 1$ for all $\delta \in (0, \delta_0)$. As a result we obtain that there exists $\gamma > 0$ such that

$$\mathbf{P}_u(\mathcal{B}^{(n)}(x\sigma r^{-1}),\ S_n \geq x\sigma - an + u) \leq Cn^{-\beta-1-\gamma}$$

for all $x \geq n^{2/3}$ where now $S_0 = u$. Setting $x\sigma = r\delta_0 an$ we get

$$\mathbf{P}_u(\mathcal{B}^{(n)}(\delta_0 an),\ S_n \geq -an/2 + u) \leq Cn^{-\beta-1-\gamma}.$$

Therefore, for every $k \in \mathbb{Z}$

$$\mathbf{P}_u(\max_{1\leq j\leq n} X_j \leq \delta_0 an;\ S_n \geq k) \leq Cn^{-\beta-1-\gamma} \tag{21}$$

for all $an/2 - u \geq M \to \infty$. Since the left-hand side is decreasing when $\delta_0 \downarrow 0$ the desired statement follows. ■

We know by (4) that for any fixed N and $l \geq -N$

$$\mathbf{P}(L_n \geq -N,\ S_n \in [l, l+1)) \sim\ b_n U(N) \int_{N+l}^{N+l+1} V(-z)\,dz,\ n \to \infty.$$

Hence, applying Lemma 9 with $u = 0$ we conclude that, as $n \to \infty$

$$\mathbf{P}(L_n \geq -N,\ S_n \in [l, l+1)) \sim \mathbf{P}\left(L_n \geq -N,\ S_n \in [l, l+1); \bar{\mathcal{B}}^{(n)}(\delta_0 an)\right),$$

meaning that for the event $\{L_n \geq -N,\ S_n \in [l, l+1)\}$ to occur it is necessary to have at least one jump exceeding $\delta_0 an$. The next statement shows that, in fact, there is exactly one such big jump on the interval $[0, n]$ that gives the contribution of order b_n to (4) and the jump occurs at the beginning of the interval.

Lemma 10 *Under conditions* $\mathbf{E}[X] = -a < 0$ *and (3) for any fixed* l *and* $\delta > 0$

$$\lim_{J\to\infty}\limsup_{n\to\infty} b_n^{-1}\mathbf{P}\left(L_n \geq -N,\ \max_{J\leq j\leq n} X_j \geq \delta an,\ S_n \in [l, l+1)\right) = 0$$

and, for any fixed J

$$\lim_{n\to\infty} b_n^{-1}\mathbf{P}\left(\cup_{i\neq j}^{J}\{X_i \geq \delta an,\ X_j \geq \delta an\}\right) = 0.$$

Proof. Write for brevity $S_n \in [l)$ if $S_n \in [l, l+1)$. Then

$$\begin{aligned}
&\mathbf{P}(L_n \geq -N, X_j \geq \delta an, S_n \in [l)) \\
&\quad\leq \int_{-N}^{\infty} \mathbf{P}(S_{j-1} \in ds, L_{j-1} \geq -N) \times \\
&\qquad\qquad \int_{\delta an}^{\infty} \mathbf{P}(X_j \in dt)\,\mathbf{P}(S_{n-j} \in [l-t-s), L_{n-j} \geq -t-s-N) \\
&\quad\leq \int_{-N}^{\infty} \mathbf{P}(S_{j-1} \in ds, L_{j-1} \geq -N) \int_{\delta an}^{\infty} \mathbf{P}(X_j \in dt)\,\mathbf{P}(S_{n-j} \in [l-t-s)).
\end{aligned}$$

By condition (3),

$$\mathbf{P}(X_j \in [t)) \leq C\frac{\mathbf{P}(X > t)}{t},\ t > 0.$$

This estimate and its monotonicity in t gives

$$\begin{aligned}
&\int_{-N}^{\infty} \mathbf{P}(S_{j-1} \in ds; L_{j-1} \geq -N) \int_{\delta an}^{\infty} \mathbf{P}(X_j \in dt)\,\mathbf{P}(S_{n-j} \in [l-t-s)) \\
\leq\ & C_1\frac{\mathbf{P}(X \geq \delta an)}{n} \int_{-N}^{\infty} \mathbf{P}(S_{j-1} \in ds; L_{j-1} \geq -N) \int_{\delta an}^{\infty} \mathbf{P}(S_{n-j} \in [l-t-s))\,dt.
\end{aligned}$$

Now

$$\begin{aligned}
\int_{\delta an}^{\infty} \mathbf{P}(S_{n-j} \in [l-t-s))\,dt \quad &\leq \quad \int_{-\infty}^{\infty} dt \int_{l-t-s}^{l-t-s+1} \mathbf{P}(S_{n-j} \in dw) \\
&= \int_{-\infty}^{\infty} \mathbf{P}(S_{n-j} \in dw) \int_{l-s-w}^{l-s-w+1} dt = 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbf{P}(L_n \geq -N, X_j \geq \delta an, S_n \in [l)) \\
&\quad\leq C\frac{\mathbf{P}(X > an)}{n} \int_{-N}^{\infty} \mathbf{P}(S_{j-1} \in ds, L_{j-1} \geq -N) \\
&\quad= C\frac{\mathbf{P}(X > an)}{n}\mathbf{P}(L_{j-1} \geq -N) = C_1 b_n \mathbf{P}(L_{j-1} \geq -N).
\end{aligned}$$

By (10) the series $\sum_{j\geq 1}\mathbf{P}(L_{j-1} \geq -N)$ converges meaning that a big jump may occur at the beginning only. Moreover, it is unique on account of the estimate

$$\mathbf{P}(X_i \geq \delta an, X_j \geq \delta an) = O\left(l^2(n)n^{-2\beta}\right) = o(b_n)$$

for all $i \neq j$ with $\max(i,j) \leq J$ and $\beta > 2$. ■

The next lemma gives an additional information about the properties of the random walk in the presence of a big jump. Let

$$\mathcal{R}_\delta(M,K) = \left\{\delta an \leq X_1 \leq an - M\sqrt{n},\ |S_n| \leq K\right\}$$

and

$$\mathcal{R}(M,K) = \left\{X_1 \geq an + M\sqrt{n},\ |S_n| \leq K\right\}.$$

Lemma 11 *Under conditions* $\mathbf{E}[X] = -a < 0$ *and (3) for any* $\delta \in (0,1)$ *and each fixed* K*,*

$$\lim_{M\to\infty} \limsup_{n\to\infty} b_n^{-1} \mathbf{P}\left(\mathcal{R}_\delta(M,K) \cup \mathcal{R}(M,K)\right) = 0.$$

Proof. Similarly to the previous lemma we have

$$\begin{aligned}
\mathbf{P}\left(\mathcal{R}_\delta(M,K)\right) &= \int_{\delta an}^{an-M\sqrt{n}} \mathbf{P}\left(S_{n-1} \in [-K-x, K-x]\right) \mathbf{P}\left(X_1 \in dx\right) \\
&\leq C\frac{\mathbf{P}(X > \delta an)}{\delta an} \int_{\delta an}^{an-M\sqrt{n}} \mathbf{P}\left(S_{n-1} \in [-2K-x, 2K-x]\right) dx \\
&= C\frac{\mathbf{P}(X > \delta an)}{\delta an} \int_{\delta an}^{an-M\sqrt{n}} dx \int_{-2K-x}^{2K-x} \mathbf{P}\left(S_{n-1} \in dv\right) \\
&\leq 4KC\frac{\mathbf{P}(X > \delta an)}{\delta an} \int_{-2K-an+M\sqrt{n}}^{2K-\delta an} \mathbf{P}\left(S_{n-1} \in dv\right) \\
&\leq 4KC\frac{\mathbf{P}(X > \delta an)}{\delta an} \mathbf{P}\left(S_{n-1} \geq -2K - an + M\sqrt{n}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{P}\left(\mathcal{R}(M,K)\right) &= \int_{an+M\sqrt{n}}^{\infty} \mathbf{P}\left(S_{n-1} \in [-K-x, K-x]\right) \mathbf{P}\left(X_1 \in dx\right) \\
&\leq C\frac{\mathbf{P}(X > an)}{an} \int_{an+M\sqrt{n}}^{\infty} \mathbf{P}\left(S_{n-1} \in [-2K-x, 2K-x]\right) dx \\
&\leq 4KC\frac{\mathbf{P}(X > an)}{an} \mathbf{P}\left(S_{n-1} \leq 2K - an - M\sqrt{n}\right).
\end{aligned}$$

Since $\limsup_{n\to\infty} \mathbf{P}\left(|S_{n-1} + an| \geq M\sqrt{n}\right)$ decreases to 0 as $M \to \infty$ by the central limit theorem, the desired statement follows. ■

5 Proof of Theorem 2

We start by the following important statement.

Lemma 12 *Let F_n be a bounded family of uniformly equicontinuous functions as defined in Theorem 2 by (6). Then the family of functions*

$$g_n(s) = \sqrt{n}\mathbf{E}_s\left[F_n(\mathbf{S}_n); L_n \geq -x, S_n \leq T\right], \quad n = 1, 2, ...,$$

is uniformly equicontinuous and uniformly bounded in $s \in \mathbb{R}$.

Proof. First, the fact that the family of functions F_n is bounded by C combined with the Stone local limit theorem for iid random variables having finite variance (see, for instance, [7], Section 8.4) allows us to bound g_n by

$$C\sqrt{n}\mathbf{P}_s(S_n \in [-x, T]) = C\sqrt{n}\mathbf{P}(S_n \in [-x - s, T - s]) \leq C_1 < \infty.$$

Second,

$$\begin{aligned}
&|g_n(s+\epsilon) - g_n(s)| \\
&\quad = \sqrt{n}|\mathbf{E}_s\left[F_n(\mathbf{S}_n + \epsilon\mathbf{1}_{n+1}); L_n + \epsilon \geq -x, S_n + \epsilon \leq T\right] \\
&\qquad\qquad -\mathbf{E}_s\left[F_n(\mathbf{S}_n); L_n \geq -x, S_n \leq T\right]| \\
&\quad \leq \sqrt{n}|\mathbf{E}_s\left[F_n(\mathbf{S}_n + \epsilon\mathbf{1}_{n+1}) - F_n(\mathbf{S}_n); L_n + \epsilon \geq -x, S_n + \epsilon \leq T\right]| \\
&\qquad\qquad + \sqrt{n}|\mathbf{E}_s\left[F_n(\mathbf{S}_n); L_n + \epsilon \geq -x, S_n + \epsilon \leq T\right] \\
&\qquad\qquad\qquad -\mathbf{E}_s\left[F_n(\mathbf{S}_n); L_n \geq -x, S_n \leq T\right]| \\
&\quad \leq H_\epsilon C + \sqrt{n}|\mathbf{P}_s(L_n + \epsilon \geq -x, S_n + \epsilon \leq T) - \mathbf{P}_s(L_n \geq -x, S_n \leq T)|,
\end{aligned}$$

where $H_\epsilon \to 0$ as $\epsilon \to 0$ again by the assumptions on F_n and the Stone local limit theorem. Let us prove now that the last term is small. Indeed,

$$\begin{aligned}
&\sqrt{n}|\mathbf{P}_s(L_n + \epsilon \geq -x, S_n + \epsilon \leq T) - \mathbf{P}_s(L_n \geq -x, S_n \leq T)| \\
&\qquad \leq \sqrt{n}\left[\mathbf{P}_s(S_n \in [T - \epsilon, T]) + \mathbf{P}_s(L_n \in [-x - \epsilon, -x[, S_n \leq T)\right]
\end{aligned}$$

and only the second term raises a difficulty. By the total probability formula with respect to the (first) time k of the minimum we have

$$\begin{aligned}
&\mathbf{P}_s(L_n \in [-x - \epsilon, -x[, S_n \leq T) \\
&\quad = \mathbf{P}(L_n + s + x \in [-\epsilon, 0[, S_n \leq T - s) \\
&\quad = \sum_{k=0}^{n}\mathbf{P}(S_1 > S_k, \cdots, S_{k-1} > S_k, S_k + s + x \in [-\epsilon, 0), \\
&\qquad\qquad\qquad S_{k+1} \geq S_k, \cdots, S_n \geq S_k, S_n \leq T - s) \\
&\quad \leq \sum_{k=0}^{n-[\sqrt{n}]-1}\mathbf{P}(S_k + s + x \in [-\epsilon, 0))\mathbf{P}(S_{k+1} \geq S_k, \cdots, S_n \geq S_k) \\
&\qquad + \sum_{k=n-[\sqrt{n}]}^{n}\mathbf{P}(S_1 > S_k, \cdots, S_{k-1} > S_k, S_k + s + x \in [-\epsilon, 0)) \\
&\qquad\qquad\qquad \times\mathbf{P}(S_{k+1} \geq S_k, \cdots, S_n \geq S_k).
\end{aligned}$$

Now we use the representation

$$\mathbf{P}(S_{k+1} \geq S_k, \cdots, S_n \geq S_k) = \mathbf{P}(L_{n-k} \geq 0) \sim C(n-k+1)^{-\beta}$$

and the Stone local limit theorem according to which

$$\sqrt{2\pi n}\mathbf{P}(S_k + s + x \in [-\epsilon, 0[) = \epsilon \exp\left\{-\frac{(s+x)^2}{2\sigma^2 n}\right\} + \delta_n,$$

where $\delta_n \to 0$ as $n \to \infty$ uniformly in $k \in [n - \sqrt{n}, n]$ and $s + x \in \mathbb{R}$. Hence we conclude that

$$\begin{aligned}
&\sqrt{n}\mathbf{P}_s(L_n \in [-x-\epsilon, -x[, S_n \leq T) \\
&\leq (\epsilon + \delta_n)\, C \sum_{k=0}^{n-[\sqrt{n}]-1} (n-k+1)^{-\beta} + C_1 \sum_{k=n-[\sqrt{n}]}^{n} (n-k+1)^{-\beta} \\
&\leq C_2 \left(\epsilon + \delta_n + \sqrt{n}(\sqrt{n})^{-\beta}\right) \leq C_3 \left(\epsilon + \delta_n\right),
\end{aligned}$$

for n large enough, since $\beta > 1$. We end up the proof by noting that all these bounds are uniform with respect to s. ■

Proof of Theorem 2. We know by Lemmas 9, 10 and 11 that conditionally on the event $\{L_n \geq -x, S_n \leq T\}$, there is a (single) big jump, that its size is of order an with a deviation of order $\sqrt{n}$ and that the jump happens at the beginning. Taking this into account and setting $\mathcal{A}_j^M = \{X_j - an \in [-M\sqrt{n}, M\sqrt{n}]\}$ we get

$$\mathbf{E}\left[F(\mathbf{S}_{j-1})F_{n-j}(\mathbf{S}_{j,n}); L_n \geq -x, S_n \leq T\right] = \varepsilon_{J,M,n} b_n + \sum_{j=0}^{J} A_{j,n}^M,$$

where

$$\lim_{J,M\to\infty} \sup_n |\varepsilon_{J,M,n}| = 0$$

and

$$A_{j,n}^M = \mathbf{E}\left[F(\mathbf{S}_{j-1})F_{n-j}(\mathbf{S}_{j,n}); L_n \geq -x, \mathcal{A}_j^M, S_n \leq T\right].$$

By the Markov property we have

$$A_{j,n}^M = \mathbf{E}\left[F(\mathbf{S}_{j-1})1_{\{L_{j-1}\geq -x\}} H_{j,n}^M(S_{j-1})\right],$$

where

$$H_{j,n}^M(s) = \mathbf{E}\left[1_{\mathcal{A}^M} \mathbf{E}_{s+X}\left[F_{n-j}(\mathbf{S}_{n-j}); L_{n-j} \geq -x, S_{n-j} \leq T\right]\right]$$

and $\mathcal{A}^M = \{X - an \in [-M\sqrt{n}, M\sqrt{n}]\}$.

STEP 1. We are proceeding by bounded convergence and show first the simple convergence. Thus, we consider

$$\begin{aligned} b_n^{-1}H_{j,n}^M(s) &= \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} b_n^{-1}\mathbf{P}(X\in dy)\mathbf{E}_{s+y}\left[F_{n-j}(\mathbf{S}_{n-j}); L_{n-j}\geq -x, S_{n-j}\leq T\right] \\ &= \frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)\mu_n(dy), \end{aligned}$$

where

$$\mu_n(dy)=b_n^{-1}\mathbf{P}(X\in dy), \quad g_{j,n}(s)=\sqrt{n}\mathbf{E}_s\left[F_{n-j}(\mathbf{S}_{n-j}); L_{n-j}\geq -x, S_{n-j}\leq T\right].$$

We want to prove that

$$\frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)\mu_n(dy)-\frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy\to 0$$

as $n\to\infty$ by using the local converges of μ_n to the Lebesgue measure (with uniformity in $y\in[an-M\sqrt{n},an+M\sqrt{n}]$ thanks to (2) and (3)) and the uniform equicontinuity of $g_{j,n}$ (compare with Lemma 12). Let us give the details. First, by Lemma 12 for any $\varepsilon>0$ there exists $\eta>0$ such that for all $n\geq n_0=n_0(\varepsilon,\eta)$ we have

$$\sup_y \sup_{u\in[0,\eta]} |g_{j,n}(y)-g_{j,n}(y+u)|\leq\epsilon.$$

Let, further, $s_i(=s_i^n)$ be a division of $[an-M\sqrt{n},an+M\sqrt{n}-1]$ into subintervals with step η. Then, for sufficiently large $n\geq n_0$,

$$\left|b_n^{-1}H_{j,n}^M(s)-\frac{1}{\sqrt{n}}\sum_i g_{j,n}(s_i)\mu_n[s_i,s_{i+1})\right|\leq 3M\epsilon.$$

Besides, $g_{j,n}(y)$ is bounded by C with respect to the pair n,y by Lemma 12. Recalling that by (3)

$$\sup_{y\in s+an+[-M\sqrt{n},M\sqrt{n}]} |\mu_n[y,y+\eta]-\eta|\leq\epsilon\eta$$

for n large enough, we get

$$\left|b_n^{-1}H_{j,n}^M(s)-\frac{1}{\sqrt{n}}\sum_i g_{j,n}(s_i)\eta\right|\leq 3M\epsilon+2CM\epsilon\eta\frac{1}{\eta}.$$

Using again the uniform continuity of $g_{j,n}$ yields for n large enough

$$\left|\frac{1}{\sqrt{n}}\sum_i g_{j,n}(s_i)\eta-\frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy\right|\leq 3\epsilon M,$$

resulting in

$$\left| b_n^{-1} H_{j,n}^M(s) - \frac{1}{\sqrt{n}} \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy \right| \leq M(6+2C)\epsilon$$

for n large enough.

Clearly, $b_n^{-1} H_{j,n}^M, n = j+1.j+2, ...$ is a bounded sequence since both $g_{j,n}$ (see Lemma 12) and $\mu_n([an - M\sqrt{n}, an + M\sqrt{n}])/\sqrt{n}$ are bounded. This and the dominated convergence theorem lead to

$$b_n^{-1} A_{j,n}^M - \mathbf{E}\left[F(\mathbf{S}_{j-1}) 1_{\{L_{j-1} \geq -x\}} \frac{1}{\sqrt{n}} \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(S_{j-1}+y)dy \right] \stackrel{n\to\infty}{\longrightarrow} 0. \tag{22}$$

STEP 2. We can now complete the proof by reversing the random walk after time j. To this aim set $\mathbf{s}_k = (s_0, \cdots, s_k)$ and $\mathbf{s}_{n,0} = (s_n, \cdots, s_0)$ and recall that (see, for instance, Lemma 9 in [14])

$$ds_0 \mathbf{P}_{s_0}(\mathbf{S}_n \in d\mathbf{s}_n) = ds_n \mathbf{P}_{s_n}(\mathbf{S}'_{n,0} \in d\mathbf{s}_{n,0}).$$

Hence, letting

$$\mathcal{B}_n(\mathbf{s}_k) = \{|s_0 - an - s| \leq M\sqrt{n},\ s_k \in [-x, T],\ \min_{0\leq i\leq k} s_i \geq -x\}$$

we get by integration

$$\begin{aligned} &\int 1_{\mathcal{B}_n(\mathbf{s}_{n-j})} F_{n-j}(\mathbf{s}_{n-j}) ds_0 \mathbf{P}_{s_0}(\mathbf{S}_{n-j} \in d\mathbf{s}_{n-j}) \\ &\quad = \int 1_{\mathcal{B}_n(\mathbf{s}_{n-j})} F_{n-j}(\mathbf{s}_{n-j}) ds_{n-j} \mathbf{P}_{s_{n-j}}(\mathbf{S}'_{n-j,0} \in d\mathbf{s}_{n-j,0}). \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{1}{\sqrt{n}} \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy \\ &= \int_{s'_0 \in [-x,T]} \mathbf{E}_{s'_0}\left[F_{n-j}(\mathbf{S}'_{n-j,0}); L'_{n-j} \geq -x; \left|S'_{n-j} - an - s\right| \leq M\sqrt{n} \right] ds'_0. \end{aligned}$$

Since, as $n \to \infty$

$$\mathbf{P}(\left|S'_{n-j} - an - s\right| \leq M\sqrt{n}) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-M}^{M} \exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy$$

for every $s \in \mathbb{R}$ and $\mathbf{S}'$ has a positive drift, we conclude that

$$\begin{aligned} K^M(s) = \limsup_{n\to\infty} \Bigg| &\frac{1}{\sqrt{n}} \int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(s+y)dy \\ &- \int_{s'_0 \in [-x,T]} \mathbf{E}_{s'_0}\left[F_{n-j}(\mathbf{S}'_{n-j,0}); L'_\infty \geq -x \right] ds'_0 \Bigg| \end{aligned}$$

goes to 0 as M becomes large. Further, by the bounded convergence and taking into account the boundness of $g_{j,n}$, we get (recall (7))

$$\limsup_{n\to\infty}\Bigg|\mathbf{E}\left[F(\mathbf{S}_{j-1})1_{\{L_{j-1}\geq -x\}}\frac{1}{\sqrt{n}}\int_{-M\sqrt{n}+an}^{M\sqrt{n}+an} g_{j,n}(S_{j-1}+y)dy\right]$$
$$-\mathbf{E}\left[F(\mathbf{S}_{j-1})1_{\{L_{j-1}\geq -x\}}\theta\mathbf{E}_\mu\left[F_{n-j}(\mathbf{S}'_{n-j,0})|L'_\infty\geq -x\right]\,\right]\Bigg|$$
$$\leq \mathbf{E}\left[F(\mathbf{S}_{j-1})1_{\{L_{j-1}\geq -x\}}K^M(S_{j-1})\right]$$

where the right-hand side goes to 0 as $M\to\infty$. Using (7) once again we set

$$D_{j,n}=\mathbf{E}\left[F(\mathbf{S}_{j-1})|L_{j-1}\geq -x\right]\mathbf{E}_\mu\left[F_{n-j}(\mathbf{S}'_{n-j})|L'_\infty\geq -x\right]$$

and deduce from (22) that the function

$$R^M=\limsup_{n\to\infty}\left|b_n^{-1}A_{j,n}^M-\mathbf{P}(L_{j-1}\geq -x)\theta D_{j,n}\right|$$

goes to zero as $M\to\infty$. Writing

$$C_{j,n}=\mathbf{E}\left[F(\mathbf{S}_{j-1})F_{n-j}(\mathbf{S}_{j,n});L_n\geq -x;\ S_n\leq T;\ X_j\geq an/2\right]$$

we have

$$\limsup_{n\to\infty}\left|b_n^{-1}C_{j,n}-\mathbf{P}(L_{j-1}\geq -x)\theta D_{j,n}\right|$$
$$\leq \limsup_{n\to\infty} b_n^{-1}\mathbf{P}(X_j\geq an/2,|X_j-an|>M\sqrt{n},\ S_n\leq T)+R^M.$$

Combining the last limit and Lemma 11 ensures that the right-hand side of this inequality goes to 0 as $M\to\infty$. We conclude

$$\lim_{n\to\infty}\left(b_n^{-1}C_{j,n}-\mathbf{P}(L_{j-1}\geq -x)\theta D_{j,n}\right)=0.$$

STEP 4. We apply the limit above to the family of functions $F=1, F_{n-j}=1$ and get

$$b_n^{-1}\mathbf{P}\left(L_n\geq -x,\ S_n\leq T,\ X_j\geq an/2\right)\overset{n\to\infty}{\longrightarrow}\mathbf{P}(L_{j-1}\geq -x)\theta. \qquad (23)$$

Recalling (4) ensures that there exists $\pi_j(x)>0$ such that

$$\mathbf{P}\left(X_j\geq an/2\mid L_n\geq -x,\ S_n\leq T\right)\overset{n\to\infty}{\longrightarrow}\pi_j(x).$$

Using Lemmas 9, 10 and 11 shows that there is only one big jump at the beginning, and it has to be greater than $an/2$. Thus, $\sum_{j\geq 0}\pi_j(x)=1$. Finally, the proof of the Theorem can be completed by using again the conclusion of STEP 3.

Acknowledgement. This work was partially funded by the project MANEGE 'Modèles Aléatoires en Écologie, Génétique et Évolution' 09-BLAN-0215 of ANR (French national research agency), Chair Modelisation Mathematique et Biodiversite VEOLIA-Ecole Polytechnique-MNHN-F.X. and the professorial chair Jean Marjoulet. The second author was also supported by the Program of the Russian Academy of Sciences "Dynamical systems and control theory".

■

References

[1] Afanasyev V.I., Boeinghoff C., Kersting G., and Vatutin V.A. (2012). Limit theorems for weakly subcritical branching processes in random environment. *J.Theor. Probab.,* **25**, N 3, pp. 703–732.

[2] Afanasyev V.I., Boeinghoff C., Kersting G., and Vatutin V.A. (2013) Conditional limit theorems for intermediately subcritical branching processes in random environment. *Ann. Inst. H. Poincaré Probab. Statist.*, In print, arXiv:1108.2127 [math.PR]

[3] Afanasyev V. I., Geiger J., Kersting G., and Vatutin V. A. (2005). Criticality for branching processes in random environment. *Ann. Probab.* **33**, pp. 645–673.

[4] Afanasyev V. I., Geiger J., Kersting G., and Vatutin V. A. (2005). Functional limit theorems for strongly subcritical branching processes in random environment. *Stoch. Proc. Appl.,* **115**, pp. 1658–1676

[5] Bansaye V., Vatutin V. (2013). On the extinction of a class of subcritical branching processes in random environment. *On arxiv.*

[6] Bertoin, J., Doney, R. A. (1996). Some asymptotic results for transient random walks. *Adv. Appl. Probab.,* **28**, pp. 207–226.

[7] Bingham N.H., Goldie C.M., and Teugels J.L. (1987). *Regular variation.* Cambridge University Press, Cambridge.

[8] Birkner M., Geiger J., and Kersting G. (2005). Branching processes in random environment - a view on critical and subcritical cases. Proceedings of the DFG-Schwerpunktprogramm *Interacting Stochastic Systems of High Complexity*, Springer, Berlin, pp. 265–291.

[9] Borovkov A.A., and Borovkov K.A. (2008). *Asymptotic analysis of random walks. Heavy-tailed distributions.* Encyclopedia of Mathematics and its Applications, 118. Cambridge University Press, Cambridge.

[10] Chover J., Ney P., and Wainger S. (1973). Functions of Probability measures. *J. Analyse Math.,* **26**, pp. 255–302.

[11] Durrett R. (1980). Conditioned limit theorems for random walks with negative drift. *Z. Wahrsch. Verw. Gebiete* 52, no. **3**, pp. 277-287.

[12] Feller W. (1971). *An Introduction to Probability Theory and Its Applications, Volume II.* John Wiley and Sons, New York.

[13] Geiger J., Kersting G., and Vatutin V.A. (2003). Limit theorems for subcritical branching processes in random environment. *Ann. I.H. Poincaré (B).* **39**, pp. 593–620.

[14] Hirano K. (1998). Determination of the Limiting Coefficient for Exponential Functionals of Random Walks with Positive Drift. *J. Math. Sci. Univ. Tokyo*, **5**, pp. 299–332.

[15] Korshunov D.A. (2006). On the distribution density of the supremum of a random walk in the subexponential case. *Siberian Mathematical Journal*, **47**, Issue 6, pp. 1060–1065

[16] Sohier J. (2010). A functional limit convergence towards brownian excursion. *Arxiv* avialable via http://arxiv.org/abs/1012.0118.

[17] Vatutin V., and Zheng X. (2012). Subcritical branching processes in random environment without Cramer condition. *Stochastic Process. Appl.*, **122**, pp. 2594–2609.